# Design Check of
# Beam mounted Motor
# Single Degree of Freedom
# Vibrational System

Richard Sweeney

MECH610 Mechanical Vibrations, Spring 2015

March 19th, 2015

Wentworth Institute of Technology

550 Huntington Ave.

Boston, MA 02115

# Contents

# Introduction

When a physical part experiences loading, it undergoes stress. When a fixed beam experiences an external loading that is normal to the direction of the beam, it experiences a moment. This moment generates a stress in the beam called bending stress. Exerting stress on an element causes the element to deform. The yield stress of a material is the amount of stress that a material can experience before permanently deforming. The Factor of Safety of a component is the ratio of yield stress to maximum stress experienced by the component. When designing an assembly, it is very important to make sure that the Factor of Safety specified by the design requirements is satisfied for each component.

A vibration is any motion that repeats itself in a cycle. Vibrations can be caused by unbalanced forces within a rotating element. Vibration systems can be simplified to 3 basic elements: the mass, the spring, and the damper. The mass stores the kinetic energy, the spring stores the potential energy, and the damper consumes the energy. Free vibration is when the system transfers its internal energy from potential energy to kinetic energy repeatedly, with no external forces. Forced vibration is when an external, repeated force acts on the system. When designing an assembly, it is important to consider all of the vibrations that will be present, and how they will affect the parts within the assembly.

For this project, a fixed beam made of AISI 1020 steel with a motor on the center is being observed. The motor spins at a speed of 1200 RPM, which causes the beam and motor to act as a vibration system. The system will be analyzed to ensure that this design has a minimum Factor of Safety of 3. The amplitude of the vibrational force, $F_0$, is 6250 N. The motor weighs 250 kg, and the weight of the beam is considered negligible for this analysis. The beam has a length, L, of 0.5m and a cross section of 30mm x 125mm (height x width). The estimated damping ratio of the beam is 0.1.

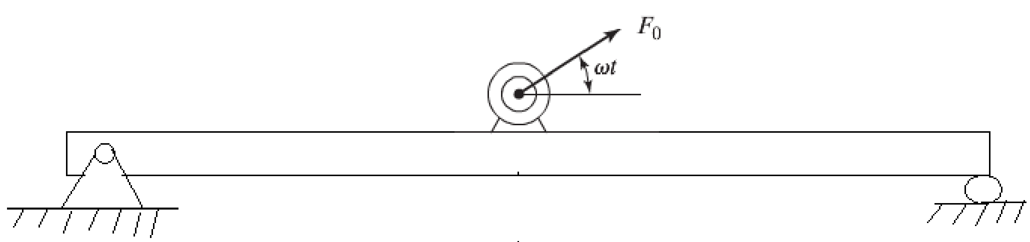



Figure 1: Beam and Motor Vibration System

## The Damped Single-Degree-Freedom Vibration Model

AISI 1020 steel has a Young's Modulus, E, of 200 GPa. Young's modulus is the ratio of stress on the material vs the resultant strain. This is important because it will be used to calculate the equivalent spring constant, k, when treating the beam as a spring. Another property of the beam that is needed to calculate the spring constant is the area moment of inertia. The equation for this is

$$I = \frac{b*h^3}{12},$$

where b is the width of the beam's cross-section, and h is the height. When the dimensions of the beam are plugged into the equation, the area moment of inertia is determined to be:

$$I = \frac{125\times10^{-3}*(30\times10^{-3})^3}{12} = 2.8125*10^{-7}\ (m^4)$$

Using this, the equivalent spring constant can now be calculated using the following equation:

$$k = \frac{48*E*I}{L^3} = \frac{48*200\times10^9*2.8125\times10^{-7}}{0.5^3} = 2.16\times10^7\left(\frac{N}{m}\right)$$

The beam and motor can be now be simplified to the following single-degree-freedom vibration system:

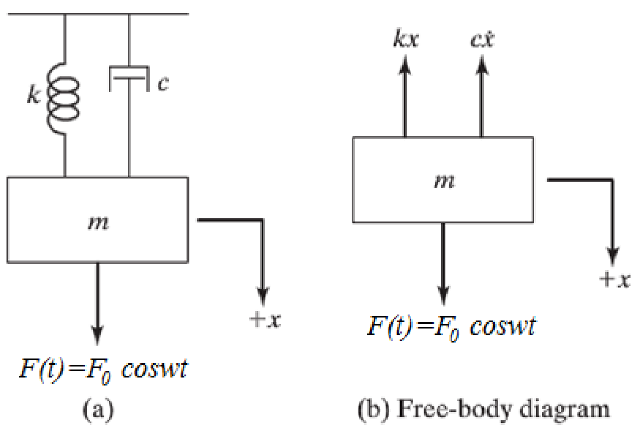



Figure 2: Single DOF vibration system

## The Governing Equation of Motion

The equation to describe the motion of the system can be derived using Newton's second law of motion, which states that the sum of all forces is equal to the mass times the acceleration. Based on the force diagram in Figure 2, we can apply Newton's second law in the x direction.

$$\Sigma F_x = ma_x$$

$$\mathrm{c} * \dot{\mathrm{x}}(\mathrm{t}) + \mathrm{k} * \mathrm{x}(\mathrm{t}) - \mathrm{F}_0 Cos(\omega t) = -m * \ddot{x}(t)$$

$$m * \ddot{x}(t) + \mathrm{c} * \dot{\mathrm{x}}(\mathrm{t}) + \mathrm{k} * \mathrm{x}(\mathrm{t}) = \mathrm{F}_0 Cos(\omega t)$$

$F_0$ is the amplitude of the external force, 6250 N. ω is the angular frequency of the external force, which can be calculated from the rotation speed of the motor.

$$\omega = 2\pi f$$

$$f = 1200\left(\frac{rotations}{minute}\right)\left(\frac{minute}{60seconds}\right) = 20\left(\frac{cycles}{second}\right)$$

$$\omega = 2\pi 20 = 40\pi\left(\frac{rad}{s}\right)$$

c is the damping constant, which can be determined from the damping ratio, ξ.

$$c = \xi * c_c$$

$$c_c = 2 * m * \omega_n$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$c = \xi * 2 * m * \sqrt{\frac{k}{m}} = 0.1 * 2 * 250 * \sqrt{\frac{2.16 \times 10^7}{250}}$$

$$c = 1469.5\left(\frac{N \times s}{m}\right)$$

## Solution of the Equation of Motion

The response of the system can has two parts, a transient response, $x_h(y)$, and a steady-state response, $x_p(t)$. The transient response is an initial response due to the initial position, $x_0$, and speed, $\dot{x}_0$, of the system when the force is applied. This response dies out quickly. The steady-state response is due to the external force, and will persist as long as the force is being applied.

$$x(t) = x_h(t) + x_p(t)$$

$$x_p(t) = X_p \cos(\omega t - \phi_p)$$

$$x_h(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0)$$

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X_p \cos(\omega t - \phi_p)$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

$$X_0 = \sqrt{\left(x_0 - X_p \cos\phi_p\right)^2 + \frac{1}{\omega_d^2}\left(\xi\omega_n x_0 + \dot{x}_0 - \xi\omega_n X_p \cos\phi_p - \omega X_p \sin\phi_p\right)^2}$$

$$\tan\phi_0 = \frac{\left(\xi\omega_n x_0 + \dot{x}_0 - \xi\omega_n X_p \cos\phi_p - \omega X_p \sin\phi_p\right)}{\omega_d\left(x_0 - X_p \cos\phi_p\right)}$$

$$X_p = \frac{F_0 / k}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}$$

$$\phi_p = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)$$

$X_p$ is the amplitude of the steady-state response, $\phi_p$ is the phase angle of the steady-state response, $X_o$ is the amplitude of the transient response, $\phi_0$ is the phase angle of the transient response, $\omega_d$ is the angular frequency of the transient response, and r is the frequency ratio.

It is assumed that the motor will be started when the beam is at rest, so $\dot{x}_0$ will be 0. To calculate $x_0$, a static analysis of the system will be done for when the motor is not running.

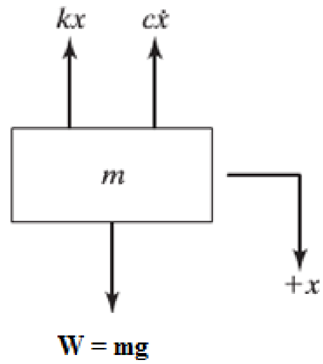



Figure 3: Static Analysis Free Body Diagram

$$\Sigma F_x = 0$$

$$c * \dot{x}_0 + k * x_0 - m * g = 0$$

$$c * 0 + k * x_0 = m * g$$

$$x_0 = \frac{m * g}{k} = \frac{250 * 9.81}{2.16 \times 10^7}$$

$$x_0 = 1.1354 \times 10^{-4} (m)$$

When the motor is first turned on, the system has an initial position of $1.135*10^{-4}$ meters due to the weight of the motor. Now the amplitudes and phase angles can be calculated:

$$r = \frac{\omega}{\omega_n} = \frac{40\pi}{293.939} = 0.427516$$

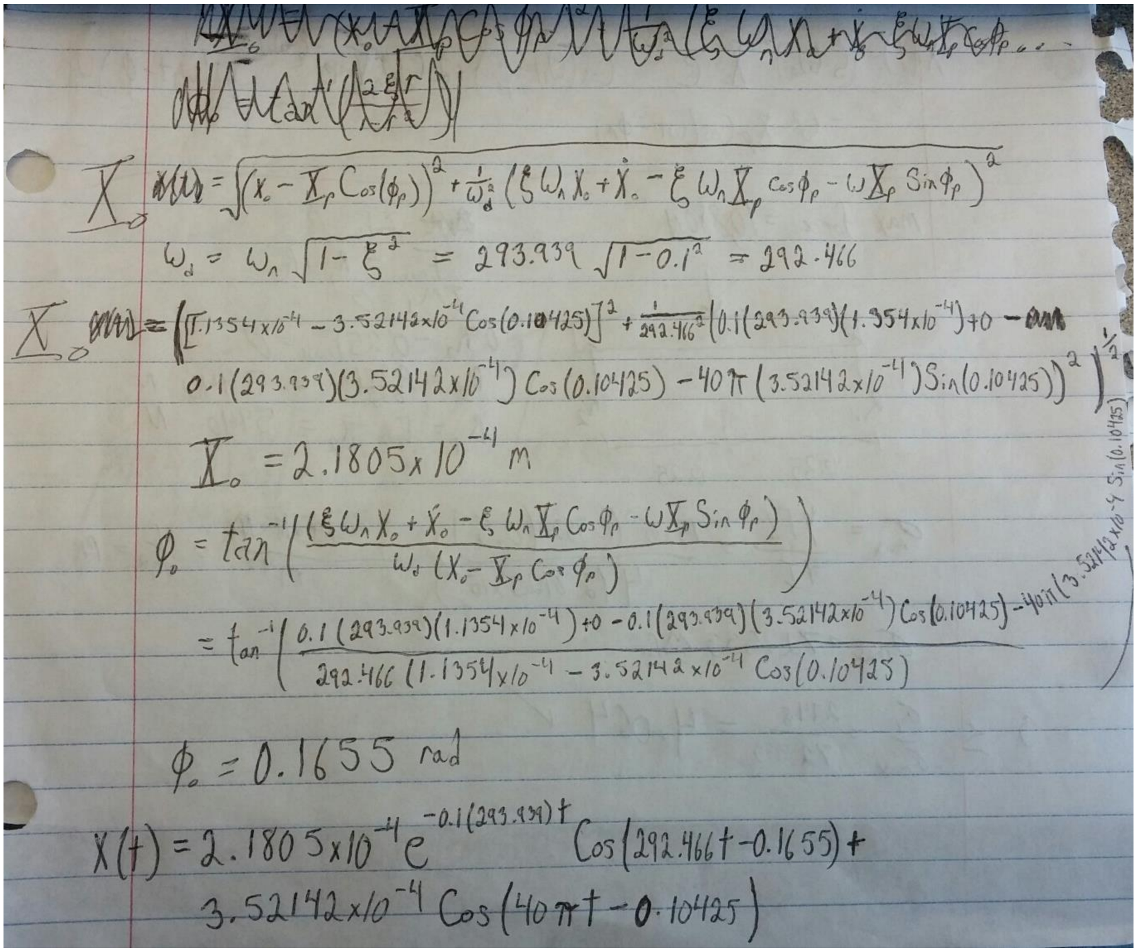
$$X_p(t) = X_p \cos(\omega t - \phi_p)$$

$$X_p = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{(6250/2.16\times10^7)}{\sqrt{(1-0.427516^2)^2 + [2(0.1)(0.427516)]^2}}$$

$$X_p = 3.52142 \times 10^{-4}\ m$$

$$\phi_p = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.1)(0.427516)}{1-0.427516^2}\right)$$

$$\phi_p = 0.10425\ rad$$



$$X_0 = \sqrt{(x_0 - X_p\cos(\phi_p))^2 + \frac{1}{\omega_d^2}(\xi\omega_n x_0 + \dot{x}_0 - \xi\omega_n X_p\cos\phi_p - \omega X_p\sin\phi_p)^2}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 293.939\sqrt{1-0.1^2} = 292.466$$

$$X_0 = \left([1.1354\times10^{-4} - 3.52142\times10^{-4}\cos(0.10425)]^2 + \frac{1}{292.466^2}\left(0.1(293.939)(1.1354\times10^{-4}) + 0 - 0.1(293.939)(3.52142\times10^{-4})\cos(0.10425) - 40\pi(3.52142\times10^{-4})\sin(0.10425)\right)^2\right)^{\frac{1}{2}}$$

$$X_0 = 2.1805\times10^{-4}\ \text{m}$$

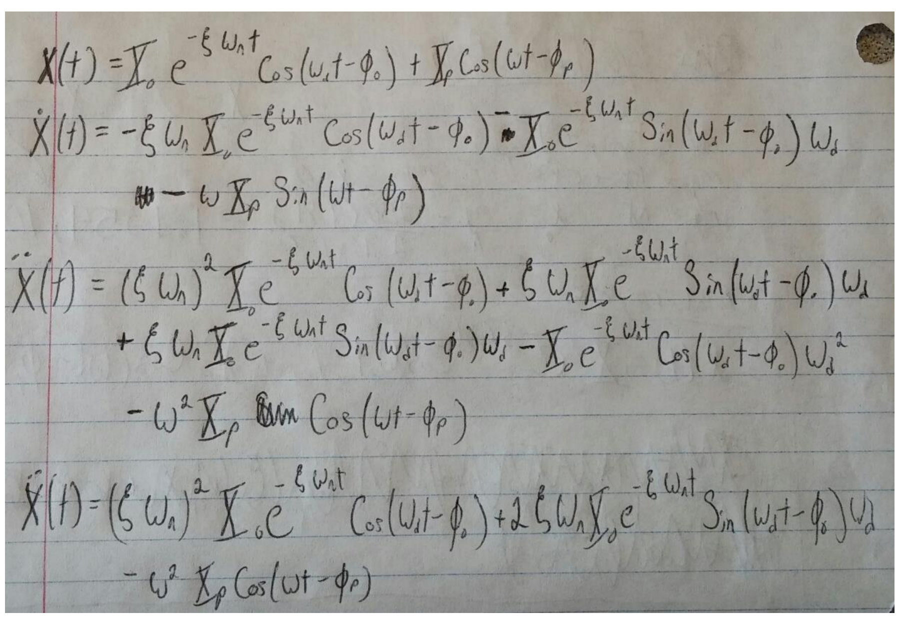
$$\phi_0 = \tan^{-1}\left(\frac{\xi\omega_n x_0 + \dot{x}_0 - \xi\omega_n X_p\cos\phi_p - \omega X_p\sin\phi_p}{\omega_d(x_0 - X_p\cos\phi_p)}\right)$$

$$= \tan^{-1}\left(\frac{0.1(293.939)(1.1354\times10^{-4}) + 0 - 0.1(293.939)(3.52142\times10^{-4})\cos(0.10425) - 40\pi(3.52142\times10^{-4}\sin(0.10425))}{292.466(1.1354\times10^{-4} - 3.52142\times10^{-4}\cos(0.10425))}\right)$$

$$\phi_0 = 0.1655\ \text{rad}$$

$$x(t) = 2.1805\times10^{-4}e^{-0.1(293.939)t}\cos(292.466t - 0.1655) + 3.52142\times10^{-4}\cos(40\pi t - 0.10425)$$

Now all of the values needed to find the response of the system are known. The equations for velocity and acceleration of the system as a function of time can be found by taking the derivative and second derivative of the equation for the positional response.



$$x(t) = X_0 e^{-\xi\omega_n t}\cos(\omega_d t - \phi_0) + X_p\cos(\omega t - \phi_p)$$

$$\dot{x}(t) = -\xi\omega_n X_0 e^{-\xi\omega_n t}\cos(\omega_d t - \phi_0) - X_0 e^{-\xi\omega_n t}\sin(\omega_d t - \phi_0)\omega_d - \omega X_p\sin(\omega t - \phi_p)$$

$$\ddot{x}(t) = (\xi\omega_n)^2 X_0 e^{-\xi\omega_n t}\cos(\omega_d t - \phi_0) + \xi\omega_n X_0 e^{-\xi\omega_n t}\sin(\omega_d t - \phi_0)\omega_d + \xi\omega_n X_0 e^{-\xi\omega_n t}\sin(\omega_d t - \phi_0)\omega_d - X_0 e^{-\xi\omega_n t}\cos(\omega_d t - \phi_0)\omega_d^2 - \omega^2 X_p\cos(\omega t - \phi_p)$$

$$\ddot{x}(t) = (\xi\omega_n)^2 X_0 e^{-\xi\omega_n t}\cos(\omega_d t - \phi_0) + 2\xi\omega_n X_0 e^{-\xi\omega_n t}\sin(\omega_d t - \phi_0)\omega_d - \omega^2 X_p\cos(\omega t - \phi_p)$$

The resultant force can now be found as a function of time by plugging these solutions into the equation of motion.

$$F(t) = m * \ddot{x}(t) + \mathrm{c} * \dot{\mathrm{x}}(\mathrm{t}) + \mathrm{k} * \mathrm{x}(\mathrm{t})$$

All of the necessary equations to analyze the response of system have been found. These equations are now be plotted in MatLAB.

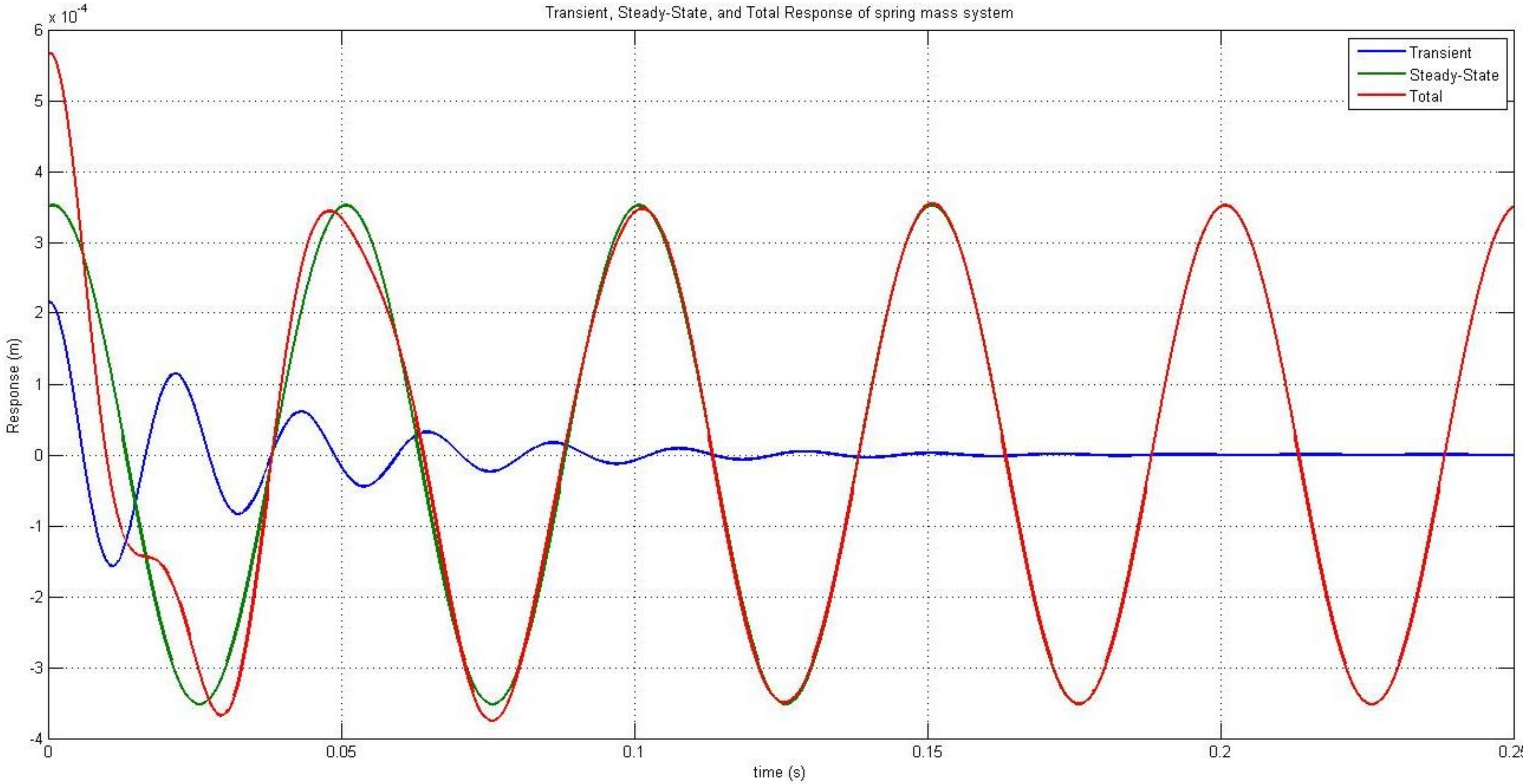



Figure 4: Positional Response of Vibrational System

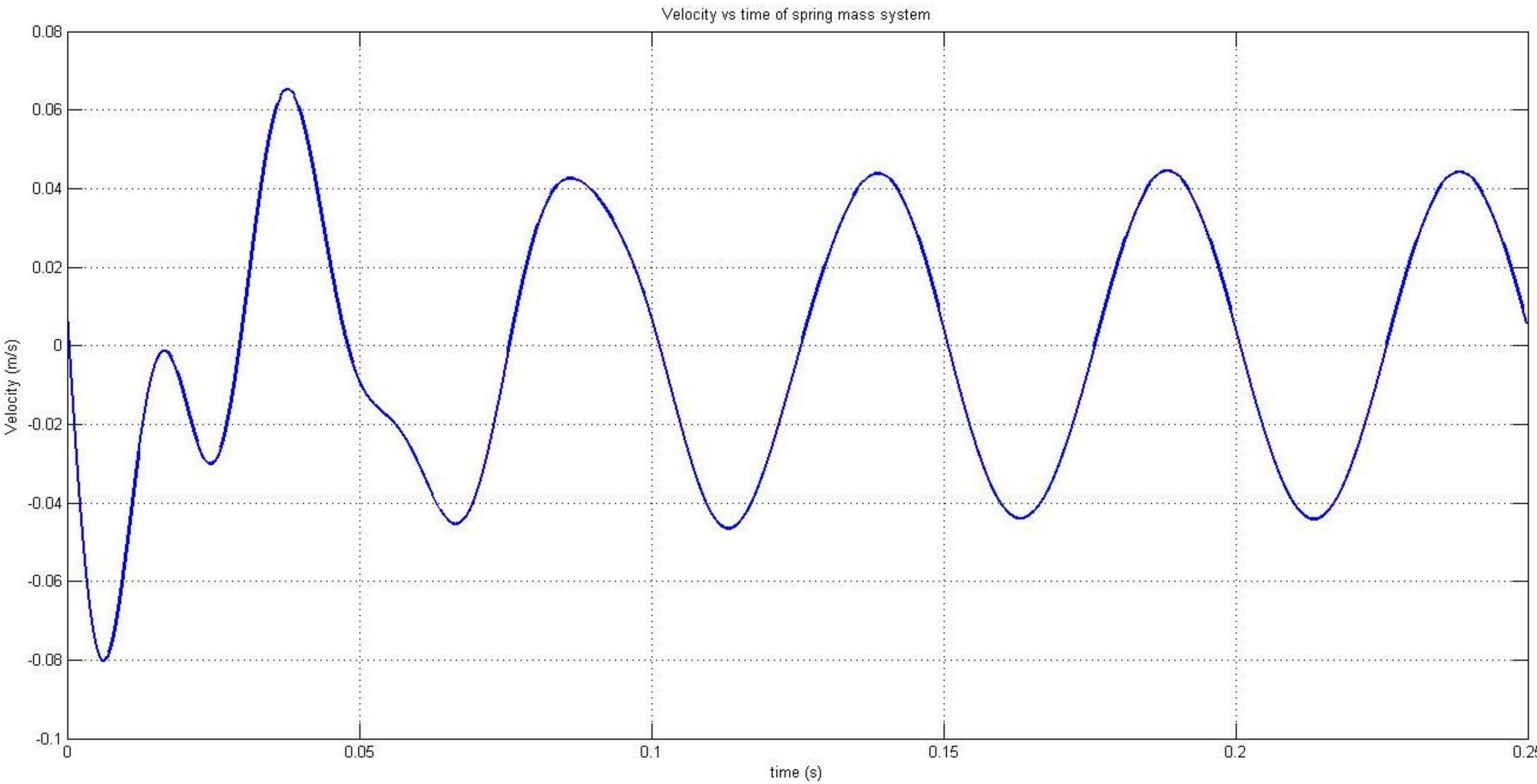



Figure 5: Velocity Response of Vibrational System

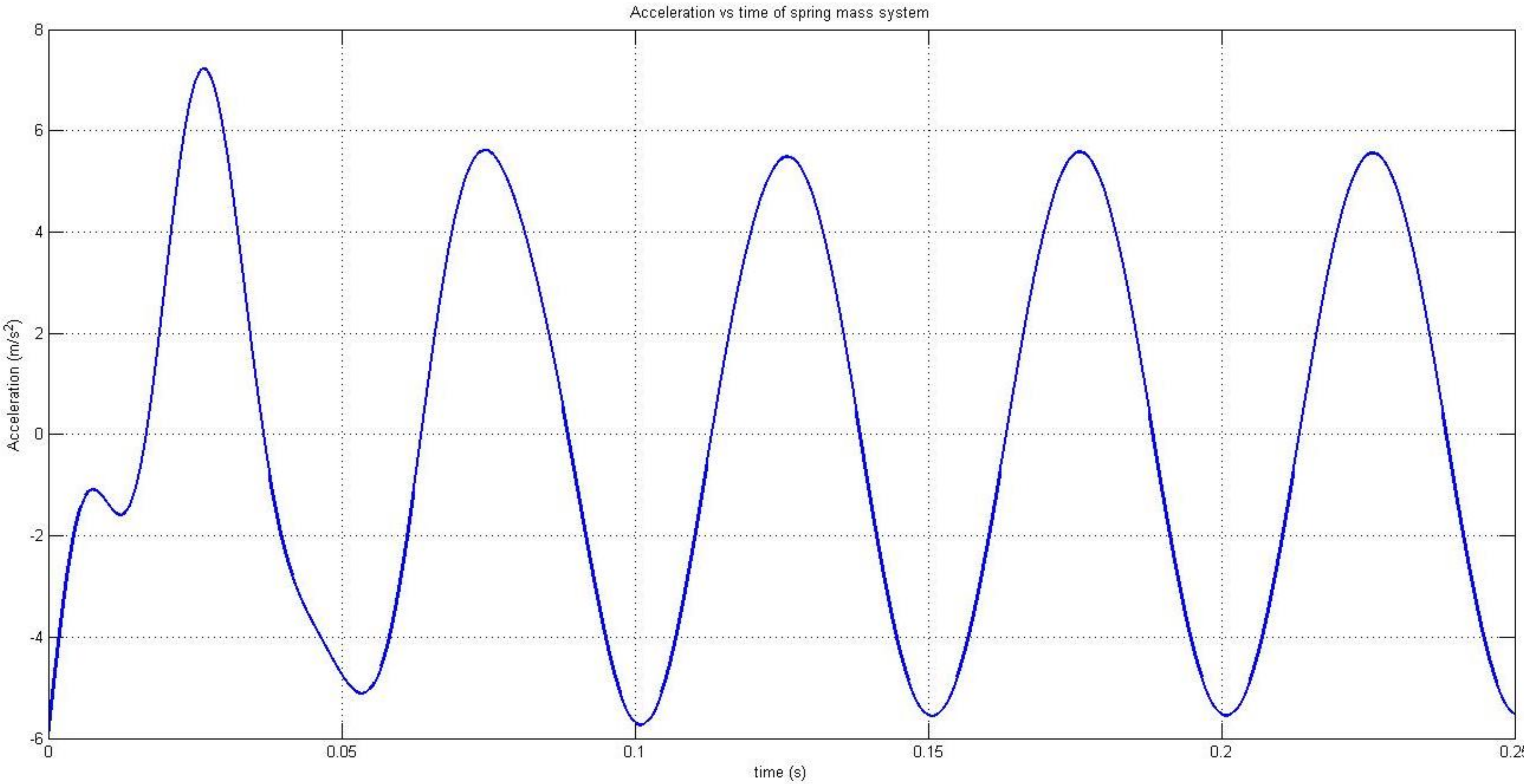



Figure 6: Acceleration Response of Vibrational System

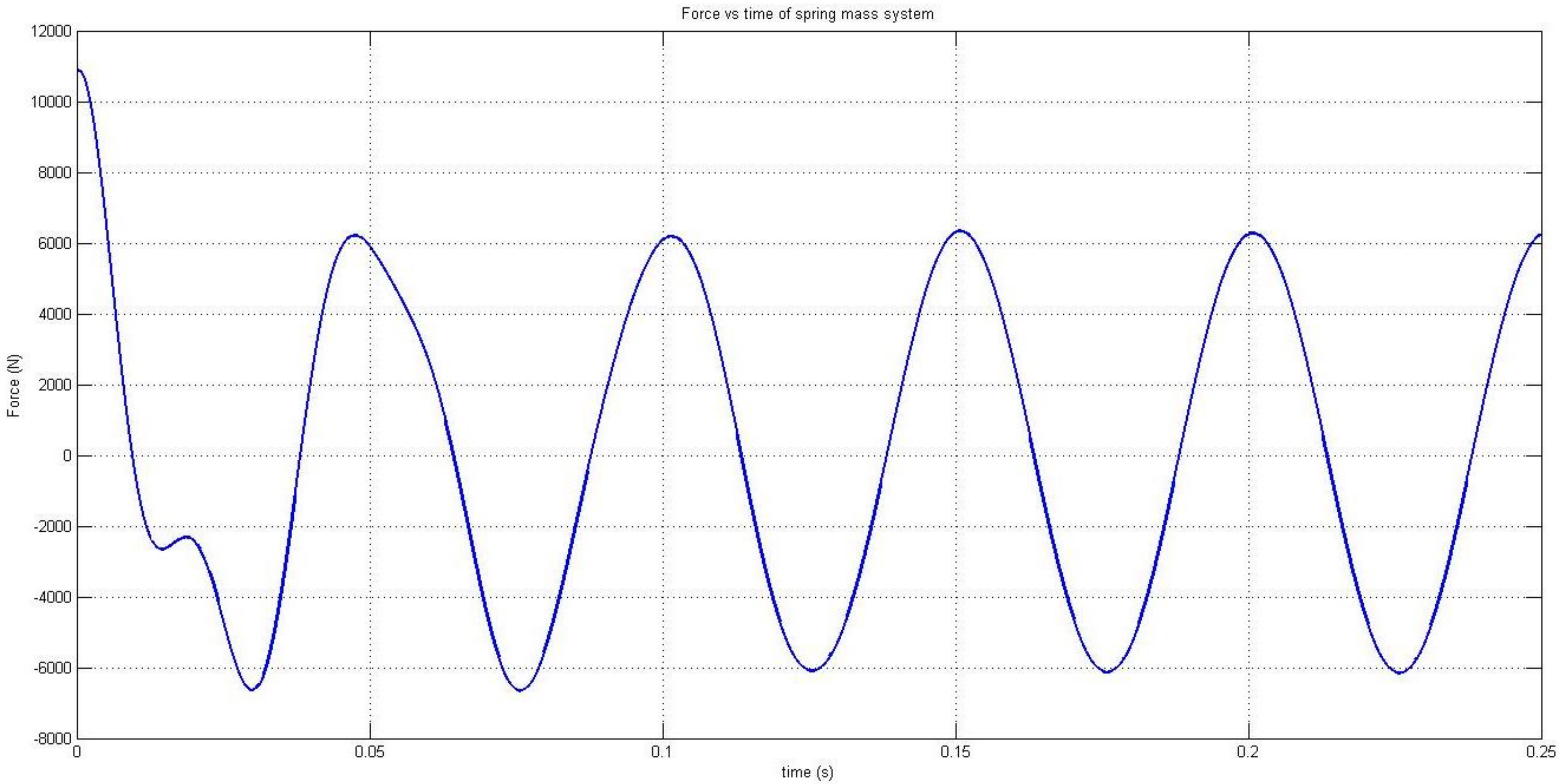
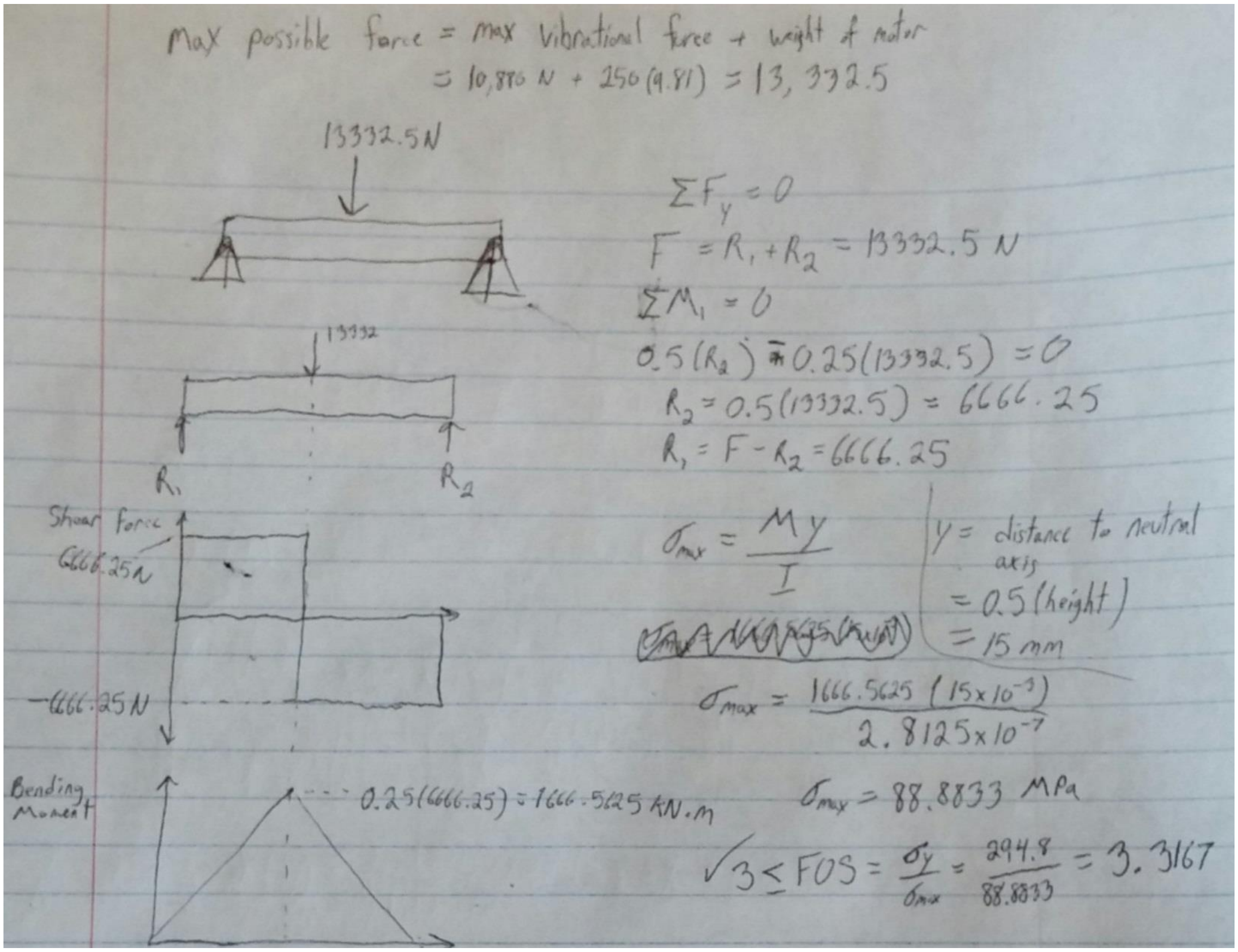



Figure 7: Force Response of Vibrational System

## Design check

Based on the graph of the resultant vibrational force in Figure 7, the maximum resultant force is 10.880 kN. The maximum force that the beam can experience is 10.880 kN plus the weight of the motor. This be the force used to run the design check. The design needs a minimum factor of safety of 3.




The minimum factor of safety is 3.3167, so the design meets the requirement.

## Discussions and Conclusions

This design should be safe to run as it is. As seen in Figure 7,when the motor is first started there is a spike in the resultant vibrational force due to the transient response, but the factor of safety is greater than 3 even at this instant. As you can see in figure 4, the transient force quickly dies out, and the overall response eventually reaches steady-state.

## References

[1] 2015, eFunda, Inc, "Properties of Carbon Steel AISI 1020," http://www.efunda.com/materials/alloys/carbon_steels/show_carbon.cfm?ID=AISI_1020&prop=all&Page_Title=AISI%201020

[2] R.C. Hibbeler, *Mechanics of Materials*, 8th Edition, Pearson Prentice Hall, 2011

## Appendixes

# The project assignment for the assessment of the ABET criteria A

Department of Mechanical and Technology

Wentworth Institute of Technology

550 Huntington Ave

Boston, MA 02115

Spring 2014

# Contents

## A: The project assignment

The supporting plate for an electrical motor has been designed. The schematic of the design is shown in the following figure 1.
An electrical motor is mounted on the mid-span of a simple-supported beam. The beam made of AISI 1020 steel has a constant cross-section 30mm X 125 mm (height X width). The span of the beam is 0.5 m. The estimation of the damping ratio is 0.1. It is assumed that the weight of the beam can be negligible. The whole system can be simplified as a damped single-degree-freedom vibration system.
The motor has a mass 250 kg with a rated speed 1200 rpm. A rotating force of magnitude $F_0 = 5200 + 50 \times roster\#(N)$ is developed due to the unbalance in the rotor of the motor. Your roster# is shown in the table 1.
It is assumed that the static failure of the beam will be the main concern and be mainly due to the bending tress and the required factor of safety for the beam is 3.0. A detailed technical analysis report is required to check whether the design is safe or not. Follow the provided the technical report format to complete the technical analysis report.
The due date for this project is: 3/5/2015. Submit the report before 5pm of 3/5/2015 through the Blackboard.

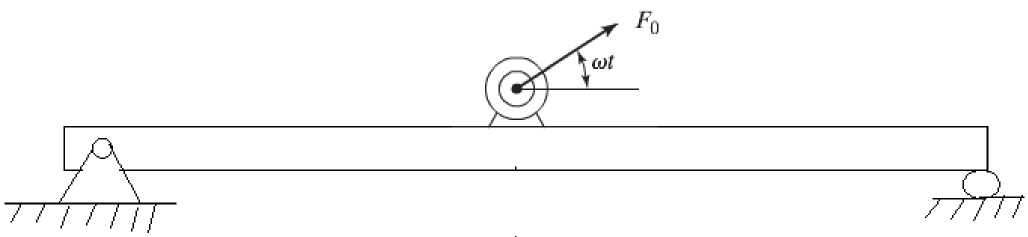



Figure1. The schematic of simple-supported beam with a motor mounted in the middle

Table 1: The roster # of MECH610-Mechanical vibration, spring 2014

| Roster # | Name | Roster # | Name |
|---|---|---|---|
| 1 | Christian A. Carey | 13 | Jonathan K. Ludwig |
| 2 | Michael A. Cushera | 14 | Richard M. Melo |
| 3 | Robert S. Drinkwater | 15 | Abraham Paredes |
| 4 | Robert A. Egan | 16 | Rutva A. Patel |
| 5 | Muhammad Fareed | 17 | Zachary D. Pearson |
| 6 | Kyle T. Finch | 18 | Greg T. Ruscak |
| 7 | Reed A. Fraser | 19 | Anthony J. Scalise |
| 8 | John M. Hinckley | 20 | Matthew S. Spaziani |
| 9 | Keith J. Hinds | 21 | Richard B. Sweeney |
| 10 | Edward J. Lariviere | 22 | Matthew J. Tacy |
| 11 | Roger A. Larrabee | 23 | Nicholas P. Timm |
| 12 | Michael J. Lindsay | 24 | |
| | | | |

## B: The report format requirement

The technical analysis report should include at least following sections and contents. You are freely to add any more sections and information.

Cover page
Table of Contents

1. Introduction
    - Some basic information about strength theory and the vibration theory.
    - The issue you are working on
2. The damped single-degree-freedom vibration model
    - The material properties of AISI 1020
    - The spring constant of the beam
    - The schematic of the damped single-degree-freedom vibration model
3. The governing equation of motion
    - The force Free-body-diagram
    - The detailed derivation process for using the Newton second law or the other approach to establish the equation of motion
4. Solution of the governing equation of motion
    - The detailed derivation process for obtaining the solution.
5. Design check
    - The shear force and bending moment diagram of the beam (it shoud include the static loading and the dynamic loading).
    - The bending stress calculation
    - The design check
6. Discussions and conclusions
    - Discussion or recommendation for the design if the design is not safe.
    - Conclusions
7. References
    - At least 2 references
8. Appendixes
    - The original assignment document
    - Some important supporting information if there is any